1. Modified wave equation:

\[ \frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2} \left[ \frac{\partial^2 E}{\partial t^2} + \omega_p^2 E \right] = 0 \]

a) Trial solution:

\[ E(z,t) = E_0 e^{i(kz - \omega t)} \]

Substitute:

\[ -k^2 E(z,t) - \frac{1}{c^2} \left[ -\omega^2 E(z,t) + \omega_p^2 E(z,t) \right] = 0 \]

So the trial solution works as long as \( \omega \) and \( k \) satisfy

\[ \omega^2 = c^2 k^2 + \omega_p^2 \]

b) The dispersion relation is as above

\[ \omega^2 = c^2 k^2 + \omega_p^2 \]

\[ w = \sqrt{c^2 k^2 + \omega_p^2} \]

\[ w = ck \]

No frequencies less than \( \omega_p \) allowed!

\[ w \to ck \text{ as } k \to \infty \]
c) To best see what is happening, solve for \( k \):

\[ k = \frac{1}{\omega} \sqrt{w^2 - w_p^2} \]

Then:

Phase velocity \( v_p = \frac{k}{ko} = \frac{c\omega}{\sqrt{w^2 - w_p^2}} = \frac{c}{\sqrt{1 - \frac{w_p^2}{w^2}}} \)

Group velocity \( v_g = \frac{c^2 k}{\omega} = \frac{c\sqrt{w^2 - w_p^2}}{\omega} = c\sqrt{1 - \frac{w_p^2}{w^2}} \)

\( v_p \cdot v_g = c^2 \) q.e.d.

When \( w > w_p \), there is a real wavevector \( k \), and the wave propagates, the phase velocity is greater than \( c \), but the group velocity is always less than \( c \).

When \( w < w_p \), the wavevector diverges and the group velocity is exactly \( c \), the wave is trapped in the plasma at its resonance frequency; the plasma takes all the energy out of the wave.

d) When \( w < w_p \), the plasma can follow the wave and the wave is "cut-off." \( k \) becomes imaginary, call it \( ik_0 \); then the wave becomes

\[ E \cos(k_0 x - \omega t) \]

The travelling wave becomes a damped oscillation or "evanescent" wave.
2. We saw in lecture that if a current \( \text{acos}(\omega t) \) was in an antenna, and the antenna was so short that the phase of the current was the same everywhere, then

\[
q_a = l \alpha \omega \cos(\omega t)
\]

However, the current cannot be so everywhere, since it must go to zero at the ends. The usual approximation is to say the current is 0 at center and \( \pm \) at ends, in which case the average is \( \frac{l\alpha}{2} \), then

\[
q_a = \frac{l\alpha}{2} \omega \cos(\omega t)
\]

Substituting this in the Larmor formula:

\[
P = \frac{q_a^2 \alpha^2}{6\pi \varepsilon_0 c^3} = \frac{\frac{l^2}{4} \alpha^2}{6\pi \varepsilon_0 c^3} \cos^2(\omega t)
\]

Convert \( \omega = 2\pi f = \frac{2\pi c}{\lambda} \). Finally

\[
P = \frac{4\pi^2}{24\pi \varepsilon_0 c} \left( \frac{1}{\lambda} \right)^2 \frac{l^2}{4} \alpha^2 \cos^2(\omega t)
\]

\[
= 20\pi^2 \left( \frac{1}{\lambda} \right)^2 \frac{l^2}{4} \alpha^2 \cos^2(\omega t)
\]

\[
P = R_{\text{rad}} \frac{l^2}{4} \alpha^2 \cos^2(\omega t)
\]

\( R_{\text{rad}} = 20\pi^2 \left( \frac{1}{\lambda} \right)^2 \), here \( f = 1 \text{ MHz} \Rightarrow \lambda = 300 \text{m} \)

If we take \( l = 40 \text{m} \), as suggested by the problem,

\[
R_{\text{rad}} = 20\pi^2 \left[ \frac{40}{300} \right]^2 \approx 3.552
\]
b) Average power

\[ \langle P \rangle = R_{\text{rad}} \frac{10^2}{2} \langle \cos^2 (\omega t) \rangle \]

\[ = R_{\text{rad}} \frac{10^2}{2} \]

\[ = R_{\text{rad}} \cdot \frac{10}{\sqrt{2}} \]

\[ I_{\text{rms}} = \frac{20}{\sqrt{2}} \]

\[ I_{\text{rms}} = \sqrt{\frac{\langle P \rangle}{R_{\text{rad}}}} = \sqrt{\frac{50 \times 10^3 \text{W}}{3.5 \Omega}} \approx 120 \text{A} \]

i. \[ \langle P \rangle = 50 \text{kw} \]

\[ I_{\text{rms}} = \sqrt{\frac{\langle P \rangle}{R_{\text{rad}}}} = \sqrt{\frac{50 \times 10^3 \text{W}}{3.5 \Omega}} \approx 120 \text{A} \]

c.) For dipole radiation,

\[ E = \frac{q \cdot a \cdot \sin \theta}{4 \pi \varepsilon_0 c^2 r} \]

Substituting for \( q \a \)

\[ E = \frac{\lambda \cdot 20 \text{W} \cdot \cos (\omega t)}{2 \pi \cdot 1 \times 10^{-6} \cdot 1} \]

\[ = \frac{20 \cdot 120 \sqrt{2} \cdot 2 \pi \cdot 1 \times 10^{-6} \cdot 1}{4 \cdot \pi \cdot (8.85 \times 10^{-12} \cdot 9 \times 10^{16} \cdot 2 \times 10^3)} \cos (\omega t) \]

\[ = E_0 \cdot \cos (\omega t) \]

\[ \Rightarrow E_0 = 1.06 \text{V/m} \]
3. Cyclotron motion:

Electron with velocity \( v \), in region of constant \( B \), moves in circles

![Cyclotron diagram](image)

\[ B = 1 \text{T} \]
\[ E_0 = 10 \text{keV} = 1.6 \times 10^{-15} \text{J} \]

\[ \text{Centripetal force is supplied by Lorentz force} \]

\[ F = \frac{mv^2}{R} = qvB \]

For electron with \( E = 10 \text{keV} = 1.6 \times 10^{-15} \text{J} \)

\[ \frac{1}{2} mv^2 = E \]
\[ v = \sqrt{\frac{2E}{m}} = \sqrt{\frac{2 \times 1.6 \times 10^{-15}}{9.1 \times 10^{-31}}} = 5.9 \times 10^7 \text{m/s} \]

\( a) \) Centripetal acceleration

\[ a = \frac{v^2}{R} = \frac{qvB}{m} = \frac{1.6 \times 10^{-19} \times 5.9 \times 10^7}{9.1 \times 10^{-31}} \]
\[ = 1.04 \times 10^{19} \text{m/s}^2 \]

\( b) \) The electron is an accelerated charge and will

induce (in pattern shown) the power according to Larmor

\[ P = \frac{q^2 v^2}{6 \pi \epsilon_0 c^3} = \frac{(1.6 \times 10^{-19})^2 (1.04 \times 10^{14})^2}{6 \pi \times 8.85 \times 10^{-12} (3 \times 108)^3} \]
\[ = \frac{6.15 \times 10^{-16} \text{W} \times 1 \text{eV}}{1.6 \times 10^{-19} \text{J}} \]
\[ = 3.8 \text{keV/sec} \]

Remember, electron starts with 10 keV, so it is losing energy first.
however, as soon as the electron loses energy, its velocity decreases, so its acceleration decreases, and the rate of energy loss decreases. how to describe this relation? put the expression for the acceleration into the larmor formula:

\[ P = \frac{q^2}{6\pi\varepsilon_0 c^3} \left[ \frac{q U R}{m} \right]^2 \]

\[ = \frac{\frac{4}{3} \pi B^2}{6\pi\varepsilon_0 c^3} \frac{m^2}{m^2} \]

\[ = \frac{\frac{4}{3} \pi B^2}{6\pi\varepsilon_0 c^3} \frac{2}{m^3} \cdot \frac{1}{2} m v^2 \]

but, \( \frac{1}{2} m v^2 = E \), and \( P = -\frac{dE}{dt} \)

\[ \frac{dE}{dt} = -\frac{\frac{4}{3} \pi B^2}{3\pi\varepsilon_0 m^3 c^3} E \]

or

\[ \frac{dE}{dt} = -\alpha E \quad \alpha = \frac{\frac{4}{3} \pi B^2}{3\pi\varepsilon_0 m^3 c^3} \]

\[ \frac{dE}{E} = -\alpha t \]

\[ E = E_0 e^{-\alpha t} \]

so, electron is damp to \( \frac{1}{e} \) of its initial energy in a time \( t = \frac{1}{\alpha} \cdot \left[ \frac{\frac{4}{3} \pi B^2}{3\pi\varepsilon_0 m^3 c^3} \right]^{-1} \approx 2.6 \text{ sec.} \)
as the electron loses energy it spirals in. To see this, just solve for the radius $R$ of the circle:

$$\frac{mv^2}{k} = qB$$

$$R = \frac{mv}{qB} = \frac{p}{qB}$$

$R$ is called the "cyclootron radius." Since $E = \frac{p^2}{2m}$, we can rewrite the radius in terms of the electron energy

$$R = \frac{\sqrt{2mE}}{qB}$$

so the radius decreases like the square root of the energy, or

$$R = \frac{\sqrt{2mE_0}}{qB} e^{-\frac{qB}{E_0}t}$$

spiraling in exponentially.
4.)

(a) Maxima when \( \Delta x = d \sin \theta = m \lambda \) 
for 2nd maximum,
\[
\sin \theta = \frac{2 \lambda}{d}
\]
\[
sin \theta \times \tan \theta = \frac{y}{L}
\]
\[
\therefore \quad \lambda = \frac{d}{2} \cdot \frac{y}{L}
\]
\[
= \frac{0.2 \times 10^{-3} \cdot 6.77 \times 10^{-3}}{2 \cdot 1.6}
\]
\[
= 4.23 \times 10^{-7} \text{ m}
\]
\[
\lambda = 4.23 \text{ nm}
\]

(b) The intensity is \( \cos^2 \left( \frac{\theta}{2} \right) \), where
\[
\frac{d \sin \theta}{\lambda} \times \frac{\pi d y}{\lambda L}
\]
the maximum is at \( \theta = 0 \), so
\[
I = 36 \%
\]
maximum when
\[
\cos^2 \left( \frac{\theta}{2} \right) = 0.36
\]
\[
\frac{\pi d y}{\lambda L} = \cos^{-1} \left[ \sqrt{0.36} \right]
\]
\[
y = \frac{\Delta L}{\pi d} \cos^{-1} \left[ \sqrt{0.36} \right]
\]
\[
= \frac{423 \times 10^{-9} \cdot 1.6}{\pi \left( 2 \times 10^{-3} \right)}
\]
\[
= 0.927
\]
\[
\approx 1 \text{ mm}.
\]
5. a) 

\[
\text{oil } n = 1.25 \\
\text{H}_2\text{O } n = 1.33
\]

There is a phase change at both reflections, so net phase change is zero and interference is entirely due to the path length difference. If constructive interference, then:

\[
2t = m \cdot \frac{\lambda}{n}
\]

For the thinnest possible layer \( m = 1 \)

\[
t = \frac{\lambda}{2n} \cdot \frac{5.25 \text{nm}}{2 \cdot 1.25} = 3.5 \times 10^{-7} \text{m} = 0.35 \mu
\]

5b) Newton's Rings.

A plano convex lens sits on a glass surface:

Light reflecting off the curved surface has no phase change. But the part of the light that reaches the glass surface and bounces back has a phase change of \( \pi \). If the distance between the two reflections is \( d \), then there will be constructive interference if the path length difference compensates for the one phase flip.
5b (cont)

\[ 2d = (m + \frac{1}{2}) \lambda \]

the rest of the problem is just to parameterize \( d \) in terms of the radial position \( r \).

if the lens has radius of curvature \( R \), then

\[ d = R - R \cos \theta = R(1 - \cos \theta) \]

for small \( \theta \), \( \cos \theta \approx 1 - \frac{1}{2} \theta^2 \)

\[ \therefore \quad d = R \left[ 1 - (1 - \frac{1}{2} \theta^2) \right] \]

\[ = \frac{1}{2} R \theta^2 \]

but if \( \theta \) is small, \( \theta \ll \theta_0 = \frac{r}{R} \), thus

\[ d = \frac{1}{2} \frac{r^2}{R} \]

substitute above:

\[ 2d = (m + \frac{1}{2}) \lambda \]

\[ 2 \cdot \frac{1}{2} \frac{r^2}{R} = (m + \frac{1}{2}) \lambda \]

\[ r = \sqrt{(m + \frac{1}{2}) \lambda R} \quad \text{q.e.d.} \]
b) the interference pattern of the two antennas can be described with the same formalism developed for 2 slit interference. turn your compass around so that the E-W axis is a vertical line, and set up as in our standard picture for two slits.

\[ \Delta x = d \sin \theta \]

At remote point \( P \), the intensity depends on the phase difference of the 2 sources at that point according to

\[ I = I_0 \cos^2 \left( \frac{\phi}{2} \right) \]

In this problem the phase difference is the result of the path length difference, as usual, and the \( \Delta \phi \) imposed on the antennas by phasing the currents. We are given that \( \Delta \phi = \frac{\pi}{2} \), but the sign is unspecified.

\[ \phi = k \Delta x + \Delta \phi \]

\[ = \frac{2\pi}{\lambda} d \sin \theta + \Delta \phi \]

We are given that \( d = \frac{d}{2} \) and \( \Delta \phi = \pm \frac{\pi}{2} \),

\[ \phi = \frac{\pi}{2} \sin \theta \pm \frac{\pi}{2} \]

\[ \frac{\phi}{2} = \frac{\pi}{4} \sin \theta \pm \frac{\pi}{4} \]

\[ I = I_0 \cos^2 \left[ \frac{\pi}{4} \sin \theta \pm \frac{\pi}{4} \right] \]

plug into calculator and plot;
<table>
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<th>$\theta$</th>
<th>$I_+$</th>
<th>$I_-$</th>
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</tr>
<tr>
<td>360</td>
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<td>0.5</td>
</tr>
</tbody>
</table>

Plot $I_+(\theta)$:

Apparently, when the eastmost antenna is $\underline{\theta}$ behind the west antenna, the power is concentrated to the west of the N-S line, as desired.