1. **Plasma Frequency**

\[
\text{area } A \rightarrow \quad \text{all charges are a distance } x \\

\text{at edges, there is a thin rectangular volume with total charge} \\

\[ q_{ \pm } = \pm eN A x \]

if \( x \) is small, we can treat these regions as surface charge densities, with \( \sigma_{ \pm } = q_{ \pm } / A = \pm e N x \).

a. Draw Gaussian pillbox as shown. by symmetry \( E \) is uniform and \( \pm \) to the sheets.

for pillbox on the right (\( - \)) side

\[ 2E \cdot a = \frac{\sigma_{ - } a}{\varepsilon_{0}} \]

\[ E = \frac{\sigma_{ - }}{2 \varepsilon_{0}} = \frac{e N x}{2 \varepsilon_{0}}, \quad \text{the sign we determine direction!} \]

the charge distribution on the left hand side will have same electric field, so that in the bulk, the total field is

\[ E = \frac{e N x}{\varepsilon_{0}} \]

b. Electron at any position in the bulk will feel this "mean field", third electron has been displaced a distance \( x \) as well. the force of the electric field on the electron will be to the left, i.e. in the \(-x\) direction, so there is a netting force on every electron

\[ F = qE = -\frac{e^2 N x}{\varepsilon_{0}} \]

c. Eqn of motion

\[ m \ddot{x} = -\frac{e^2 N x}{\varepsilon_{0}} \]

\[ \rightarrow \]
\[ \ddot{X} + \frac{e^2 N}{m \bar{v}^2} X = 0 \]

s.h.o. \[ \ddot{X} + \omega_0^2 X = 0 \quad \omega_0 = \sqrt{\frac{e^2 N}{m \bar{v}^2}} \]

The positive and negative charges oscillate back and forth with respect to each other with \( \omega_0 \), the plasma frequency.

d. For F2 layer of ionosphere: \( N = 10^{12}/m^3 \)

\[ \omega_0 = \sqrt{\frac{1.6 \times 10^{-9} \text{C}^2}{1.6 \times 10^{-31} \text{J} \cdot\text{s}}} \]

\[ = \frac{1}{9.1 \times 10^{-31} \text{J} \cdot\text{s}} \times 8.35 \times 10^{-12} \text{C}^2 \text{N}^{-1} \text{m}^{-2} \]

\[ = 56.4 \text{ MHz} \]

\[ f = \frac{\omega_0}{2\pi} = 8.9 \text{ MHz} = \text{short wave radio} \]
2. Kirchhoff:

\[ V = iR \quad \text{and} \quad i = CV \]

\[ V = L \frac{di}{dt} \]

\[ L \frac{di}{dt} + iR + \frac{q}{C} = V_{\text{res}}(\omega t) \]

a.) \[ \frac{d^2q}{dt^2} + \frac{1}{LC} \frac{dq}{dt} + \omega^2 q = \frac{V_0 e^{i\omega t}}{L} \]

Now, go to complex variables:

\[ V_{\text{res}}(\omega t) = \text{Re} \left[ V_0 e^{i\omega t} \right] \]

b.) Final solution:

\[ q = q_0 e^{i(\omega t + \phi)} \]

(assume, in steady state, that system is driven at same frequency \( \omega \) force, with unknown amplitude \( q_0 \) and phase delay \( \phi \))

\[ \dot{q} = i\omega q_0 e^{i(\omega t + \phi)} \]

\[ \dot{q} = -\omega^2 q_0 e^{i(\omega t + \phi)} \]

\[ -\omega^2 q_0 e^{i(\omega t + \phi)} + i\omega R q_0 e^{i(\omega t + \phi)} + \omega_0^2 q_0 e^{i(\omega t + \phi)} + \frac{V_0 e^{i\omega t}}{L} \]

forms:

\[ a + i \cdot b = \text{magnitude-phase} \]

modulus:

\[ \sqrt{(\omega_0^2 - \omega^2) q_0^2 + \omega^2 \omega_0^2 q_0^2 + \omega_0^2 q_0^2 + \frac{V_0^2}{L}} \]

\[ \Rightarrow q_0(\omega) = \frac{V_0/L}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \omega_0^2}} \]

phase:

\[ \tan \phi = \frac{\omega R}{\omega_0^2 - \omega^2} \]
so, finally the charge on the capacitor is given by the real part of the final column,

\[ q = \frac{V_0 L}{\sqrt{(w_0^2 - w^2)^2 + 4w^2y^2}} \cos(wt - \phi) \quad \phi = \tan^{-1}\left[\frac{wY}{w_0^2 - w^2}\right] \]

c.

\[ i(w) = \frac{dq}{dw} = \frac{-(V_0 L)w}{\sqrt{(w_0^2 - w^2)^2 + 4w^2y^2}} \sin(wt - \phi) \]

to find the resonant frequency, find where \(|i(w)|\) is maximum:

\[ \frac{di(w)}{dw} = 0 \]

a little algebra suffices to show that \(\omega_{res} = \omega_0\) at this frequency

\[ i(\omega_0) = \frac{-V_0 L}{\omega_0} \frac{\omega_0}{Y} \sin(\omega_0 t - \phi) \quad \phi = \tan^{-1}\left[\frac{wY}{w_0^2 - w_0^2}\right] = \frac{\pi}{2} \]

since \(Y = \frac{R}{L}\), and \(\phi = \frac{\pi}{2}\), we have

\[ i(\omega_0) = -\frac{V_0}{R} \sin(\omega_0 t - \frac{\pi}{2}) \]

\[ = \frac{V_0}{R} \cos(\omega_0 t) \]

since the driving voltage was \(V_0 \cos(\omega t)\), see that the current is completely in phase with driver at resonance.

reason: at \(\omega = \omega_0\), circuit is being driven at natural frequency of the LC oscillation, there is no additional "reactance" due to the L and C, and the circuit responds like a simple resistor.
3. French 5-6

\[
\begin{align*}
&k \quad A \quad k \\
\downarrow m &\quad -m &\quad -m &\quad \rightarrow B \quad -m &\quad \rightarrow \quad k \\
&k \cdot m \\
\frac{m_1 \cdot m_2}{m_3} \quad \text{if one mass is clamped, the other is like this:} \\
&k \quad m \\
\downarrow m &\quad -m &\quad \rightarrow x
\end{align*}
\]

\[m \ddot{x} = -kx - kx = -2kx\]

\[\ddot{x} + \frac{2k}{m} x = 0 \implies \omega = \sqrt{\frac{2k}{m}}\]

Then, since we are given the period in this case, \(T_c = 3\text{ sec}\), we know that \(k\) and \(m\) satisfy

\[T_c = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{2k}} = 3\text{ sec}.
\]

Now, the problem at hand is the coupled system:

\[
\begin{align*}
&m \ddot{x}_A = -kx_A - k(x_A - x_B), \\
&m \ddot{x}_B = -kx_B - k(x_B - x_A)
\end{align*}
\]

Add to get

\[
\begin{align*}
&m(x_A + x_B) + k(x_A + x_B) = 0 \\
&\ddot{q}_1 + \omega_1^2 q_1 = 0 \\
&q_1 = x_A + x_B, \quad \omega_1 = \sqrt{\frac{k}{m}}
\end{align*}
\]

Subtract to get

\[
\begin{align*}
&m(x_A - x_B) + 3k(x_A - x_B) = 0 \\
&\ddot{q}_2 + \omega_2^2 q_2 = 0 \\
&q_2 = x_A - x_B, \quad \omega_2 = \sqrt{\frac{3k}{m}}
\end{align*}
\]

\[
\begin{align*}
T_1 &= 2\pi \sqrt{\frac{m}{k}} \\
T_1 &= \frac{2\pi \sqrt{\frac{m}{k}}}{2\pi \sqrt{\frac{m}{12k}}} = \sqrt{2} \\
\frac{T_1}{T_0} &= \frac{2\pi \sqrt{\frac{m}{k}}}{2\pi \sqrt{\frac{m}{12k}}} = \sqrt{2} \\
&\implies T_1 = \sqrt{2} T_0 = 3\sqrt{2}
\end{align*}
\]

\[
\begin{align*}
T_2 &= 2\pi \sqrt{\frac{k}{3k}} \\
T_2 &= \frac{2\pi \sqrt{\frac{m}{3k}}}{2\pi \sqrt{\frac{m}{12k}}} = \frac{\sqrt{2}}{3} \\
&\implies T_2 = \frac{\sqrt{2}}{3} T_0 = \sqrt{\frac{2}{3}} \cdot 3 = \sqrt{6}
\end{align*}
\]
With these w's the normal modes are then

\[ q_1 = A_1 \cos(w_1 t + \varphi), \quad q_2 = A_2 \cos(w_2 t + \varphi) \]

\( q_1 \) is the symmetric mode, \( q_2 \) is antisymmetric.

For a particular motion, one re-express \( x_1 \) and \( x_2 \) in terms of the \( q \)'s:

\[ x_A = \frac{1}{2}(q_1 + q_2) = \frac{1}{2}(A_1 \cos(w_1 t + \varphi) + A_2 \cos(w_2 t + \varphi)) \]

\[ x_B = \frac{1}{2}(q_1 - q_2) = \frac{1}{2}(A_1 \cos(w_1 t + \varphi) - A_2 \cos(w_2 t + \varphi)) \]

b. \( B \) is pulled aside a distance \( B_0 \) (cm) and released. \( A \) is at equilibrium. Initial velocities are 0. We use these initial conditions to solve for \( A_1, A_2, \varphi_1, \varphi_2 \):

\[ \text{vel.} \Rightarrow x_{A_1} = \frac{1}{2}(A_1 \sin(w_1 t + \varphi_1) + A_2 \sin(w_2 t + \varphi_2)) = 0 \]

\[ \text{vel.} \Rightarrow x_{A_2} = \frac{1}{2}(A_1 \sin(w_1 t + \varphi_1) - A_2 \sin(w_2 t + \varphi_2)) = 0 \]

\[ x_A = 0 \Rightarrow A_1 + A_2 = 0 \Rightarrow A_1 = -A_2 \]

\[ x_B = B_0 \Rightarrow \frac{1}{2}(A_1 - A_2) = B_0 \Rightarrow A_1 = B_0, \quad A_2 = -B_0 \]
So, finally

\[ \begin{align*}
X_A &= \frac{1}{2} B_0 \left[ \cos(\omega_1 t + \phi_1) - \cos(\omega_2 t + \phi_2) \right] \\
X_B &= \frac{1}{2} B_0 \left[ \cos(\omega_1 t + \phi_1) + \cos(\omega_2 t + \phi_2) \right]
\end{align*} \]

This is the addition or subtraction of vibrations of different frequencies, which we know leads to the formula for beats.

Going through the trig, we find

\[ \begin{align*}
X_A &= B_0 \left[ \sin \left( \frac{\omega_1 - \omega_2}{2} t \right) + \sin \left( \frac{\omega_1 + \omega_2}{2} t \right) \right] \\
X_B &= B_0 \left[ \cos \left( \frac{\omega_1 - \omega_2}{2} t \right) \left[ \cos \left( \frac{\omega_1 + \omega_2}{2} t \right) \right] \right]
\end{align*} \]

At \( t = 0 \), \( X_A = 0 \) and \( X_B = B_0 \), as desired.

The time development of \( X_A \) and \( X_B \) look like this:

The time to transfer the woman from B to A and back is \( \frac{1}{2} \) the period of the beat envelope.

\[
\frac{1}{2} T_B = \frac{1}{2} \frac{2\pi}{\omega_2 - \omega_1} = \frac{2\pi}{\omega_2 - \omega_1} \cdot \frac{2\pi}{2\pi \left[ \frac{T_2}{T_1} - \frac{T_1}{T_2} \right]} = \frac{T_2 T_1}{T_1 - T_2}
\]
The situation will only be totally restored if $R_0$ is back to $R_0$, and $x_4$ is back to 0, that is, if the time above is an integer number of burn periods. It is an integer number of rather, and multiplying it by any number of times does not make it so. But because of the minimal number of the burn never repeats.
a. \[ \frac{\partial^2 \psi}{\partial y^2} - \frac{1}{N^2} \frac{\partial^2 \psi}{\partial t^2} = 0 \quad \psi = \frac{1}{N} \]

the travelling wave solution:

\[ \psi(x,t) = A \sin (kx - \omega t) + B \cos (kx - \omega t) \]

\[ \frac{\partial^2 \psi}{\partial x^2} = -k^2 \psi(x,t) \quad \text{Subsitute} \]
\[ \frac{\partial^2 \psi}{\partial t^2} = -N^2 \psi(x,t) \quad \Rightarrow \quad -k^2 \psi + \frac{\omega^2}{N^2} \psi = 0 \]

so \( \psi \) is a solution provided

\[ \omega^2 = k^2 N^2 \]

this is the "certain k and \( \omega \)."

b. \[ \psi(x,t) = \psi(x) \cos (\omega t) \]

Substitute

\[ \frac{\partial^2 \psi}{\partial x^2} + \frac{\omega^2}{N^2} \psi(x) = 0 \]

so, \( \psi(x,t) \) is a solution, as long as \( \psi(x) \) satisfies

\[ \frac{\partial^2 \psi(x)}{\partial x^2} + k^2 \psi(x) = 0 \quad k = \frac{\omega}{N} \]

c. \[ \psi(x,t) \bigg|_{t=0} = 0 \]

\[ \psi(x,t) \bigg|_{x=L} = 0 \]

d. the general solution to \[ \frac{\partial^2 \psi}{\partial x^2} + k^2 \psi = 0 \]

is

\[ \psi(x) = A \cos (kx) + B \sin (kx) \]

impose boundary conditions:

\[ \psi(0) = 0 \Rightarrow A = 0 \]

\[ \psi(L) = 0 \Rightarrow k = \frac{n \pi}{L} \quad n = 1, 2, 3, \ldots \]

\[ \lambda = \frac{2\pi}{k} = \frac{1}{n}(2L) \]

\[ \psi(x) = B \sin \left( \frac{n \pi}{L} x \right) \]
The solutions are sine waves with wavelength \( \lambda \) such that an integral number of half wavelengths fit into \([0, L]\) satisfying the boundary conditions:

\[
\begin{align*}
\lambda &= 2L \\
\lambda &= L \\
\lambda &= \frac{2}{3}L \\
\lambda &= \frac{1}{2}L
\end{align*}
\]

The lowest frequency mode has:
\[
f_1 = \frac{1}{2L} \sqrt{\frac{\pi}{m}}
\]

The higher modes are all multiples, or "higher harmonics," of the fundamental \( f_1 \).
5. Use problem #4 results.

i) If we assume the string is vibrating in fundamental mode, then

\[ \lambda_1 = 2L \]

\[ f_1 = \frac{1}{2L} \sqrt{\frac{T}{\mu}} \]

\[ \Rightarrow T = 4f_1^2 L^2 \mu \]

In the problem, we are given \( \rho : \frac{\text{mass}}{\text{volume}} \) and the diameter \( d \), so we can calculate the linear mass density \( \mu \):

\[ \frac{A}{\pi} = \frac{\rho}{\mu} \]

\[ M = \rho \cdot l \cdot A \]

\[ \mu = \frac{M}{l} = \rho \cdot A = \rho \pi \cdot \left( \frac{d}{2} \right)^2 \]

\[ \Rightarrow T = 4f_1^2 L^2 \rho \pi \left( \frac{d}{2} \right)^2 \]

for the given parameters:

\[ T = 4 \cdot \left( \frac{24.4}{5} \right)^2 \cdot (0.64 \text{ cm})^2 \cdot 7.8 \times 10^3 \text{ kg/m}^3 \cdot \pi \cdot \left( \frac{178 \times 10^{-3} \text{ m}}{2} \right)^2 \]

\[ = 76.9 \text{ kg} \cdot \text{m/s}^2 \]

\[ = 76.9 \text{ N} \]

ii) Middle \( C = 26.2 \text{ cm}^3 \)

\[ f \propto \frac{1}{L} \Rightarrow \frac{f_2}{f_1} = \frac{L_1}{L_2} \]

\[ \therefore L_2 = L_1 \frac{f_1}{f_2} = 64 \text{ cm} \cdot \frac{24.4}{26.2} = 60.1 \text{ cm} \]

It must start at 4 cm from one end.