1. **Simple harmonic motion** implies the functional form
   \[ x(t) = A \sin(\omega t + \delta) \]
   
   a) The stated "maximum excursion" of 10 cm gives \( A = 10 \text{ cm} \) while \( \omega \) and \( \delta \) are determined from the boundary conditions
   \[ x(0) = 0 = A \sin \delta \Rightarrow \delta = 0 \]
   \[ x'(0) = 40 \text{ cm/s} = A \omega \Rightarrow \omega = 4 \text{ rad/s} \]
   
   And now we are in position to derive
   
   Frequency: \( f = \frac{\omega}{2\pi} = \frac{2}{\pi} \text{ Hz} \)
   
   Period: \( T = f^{-1} = \frac{\pi}{2} \text{ s} \)
   
   Spring Constant: \( k = m\omega^2 = 8 \text{ kg} / \text{s}^2 \)
   
   b) The completely specified function of the position of the block vs. time is then
   \[ x(t) = 10 \text{ cm} \cdot \sin \left(\frac{4t}{\sec}\right) \]
   \[ v(t) = v'(t) = 40 \text{ cm/s} \cdot \cos \left(\frac{4t}{\sec}\right) \]

   c) \( x' \left( \frac{\pi}{4} \right) = 40 \text{ cm/s} \cdot \cos \pi = -40 \text{ cm/s} \). At this point, block has returned to the origin, but now is travelling in the opposite direction to complete the second half of its cycle.

   d) **Kinetic energy:**
   \[ K(t) = \frac{1}{2}mv^2 = \frac{1}{2} \cdot 0.5 \text{ kg} \cdot (0.4 \text{ m/s})^2 \cdot \cos^2 4t \]
   \[ = 0.04 \cos^2 4t \text{ kg m}^2 / \text{s}^2 \text{ (Joule)} \]
   
   **Potential energy:**
   \[ U(t) = \frac{1}{2}kx^2 = \frac{1}{2} \cdot 8 \text{ kg/s}^2 \cdot (0.1 \text{ m})^2 \sin^2 4t \]
   \[ = 0.04 \sin^2 4t \text{ kg m}^2 / \text{s}^2 \]
   
   Joules

   ![Graph of K and U vs. t](image)

   Note that the sum \( K + U = 0.04(\cos^2 4t + \sin^2 4t) = 0.04 \text{ J} \) is constant. See g)

   Since this block/spring system is identical to the previous one we can carry over the circular frequency \( \omega \), since it is completely determined by the physical properties of the oscillating system and not by initial conditions. Here, they are
   \[ x(t = 0) = 10 \text{ cm} \Rightarrow A \sin \delta = 10 \]
   \[ x'(t = 0) = 0 \text{ cm/s} \Rightarrow \cos \delta = 0 \]
   
   \[ A = 10 \text{ cm}, \ \delta = \frac{\pi}{2} \]
   
   Therefore the equation of motion is \( x(t) = 10 \text{ cm} \sin(\omega t + \frac{\pi}{2}) \).
Phase difference between two systems is $\frac{\pi}{2}$.

The total energy is the sum of the kinetic and potential.

$$E_{TOT} = \frac{1}{2} m\ddot{x}^2 + \frac{1}{2} kx^2 = 0.04 \left[ \cos^2 (4t + \frac{\pi}{2}) + \sin^2 (4t + \frac{\pi}{2}) \right] = 0.04 \, \text{J}.$$

Identical to the energy of the first system. Because the two systems are equal up to a phase difference, intuitively we expect the total energies to be the same, at least in amplitude. However, here the total energy is time-independent so then it must be identical for both systems. In fact, the time independence of the total energy is "seeded" in the differential equation:

$$\frac{\partial E_{TOT}}{\partial t} = \frac{\partial}{\partial t} \left( \frac{1}{2} m\ddot{x}^2 + \frac{1}{2} kx^2 \right)$$

$$= m\ddot{x} + k\dddot{x}$$

$$= \dddot{x} (m\ddot{x} + kx), \text{ but, by diff. eqn. of motion, } m\ddot{x} = -kx, \text{ thus,}$$

$$\frac{\partial E_{TOT}}{\partial t} = 0$$
(2) \[ z_1 = a + bi \]
\[ z_2 = c + di \]

\[ \Rightarrow z = z_1 \cdot z_2 = (a + bi)(c + di) \]
\[ = (ac - bd) + (cb + ad)i \]

**Graphically:**

![Graphical representation of complex numbers](image)

\[ z = z_1 \cdot z_2 \]

a. **Prove** \[ |z| = |z_1| \cdot |z_2| \]

\[ |z_1| = \sqrt{a^2 + b^2} \]
\[ |z_2| = \sqrt{c^2 + d^2} \]

\[ |z| = \sqrt{(ac - bd)^2 + (cb + ad)^2} \]
\[ = \sqrt{a^2c^2 + b^2d^2 + c^2b^2 + a^2d^2} \]
\[ = \sqrt{(a^2 + b^2)(c^2 + d^2)} \]
\[ = |z_1| \cdot |z_2| \quad \text{Q.E.D.} \]

b. **Prove** \[ \theta_3 = \theta_1 + \theta_2 \]
we have:

\[ \cos \theta_1 = \frac{a}{|z_1|} \quad \sin \theta_1 = \frac{b}{|z_1|} \]
\[ \cos \theta_2 = \frac{c}{|z_2|} \quad \sin \theta_2 = \frac{d}{|z_2|} \]
2b (cont.):  
Consider the sine and cosine of the sum \( \theta_1 + \theta_2 \):

\[
\cos(\theta_1 + \theta_2) = \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2
\]
\[
= \frac{a}{|z_1|} \cdot \frac{c}{|z_2|} - \frac{b}{|z_1|} \cdot \frac{d}{|z_2|}
\]
\[
= \frac{ac - bd}{|z_1| |z_2|}
\]

\[
\sin(\theta_1 + \theta_2) = \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2
\]
\[
= \frac{bc + ad}{|z_1| |z_2|}
\]

Further, the tangent of the angle \( \theta_1 + \theta_2 \) is

\[
\tan(\theta_1 + \theta_2) = \frac{\sin(\theta_1 + \theta_2)}{\cos(\theta_1 + \theta_2)}
\]
\[
= \frac{bc + ad}{ac - bd}
\]

Now, from the diagram for the number \( z = z_1 z_2 \), we also see that

\[
\tan(\theta_2) = \frac{cb + ad}{ac - bd}
\]

Convinced with above, we see \( \theta_2 = \theta_1 + \theta_2 \) Q.E.D.

C. in exponential notation

\[
|z_1| = |z_1| e^{i\theta_1}
\]
\[
\quad \text{z}_1 \text{ and } \theta_1 \text{ as previous page}
\]
\[
|z_2| = |z_2| e^{i\theta_2}
\]
\[
\quad \text{z}_2 \text{ and } \theta_2 \text{ as previous page}
\]

\[
|z| = |z_1||z_2| e^{i(\theta_1 + \theta_2)} = |z_1| e^{i\theta_1} |z_2| e^{i\theta_2}
\]
\[
|z_1| = |z_1||z_2| \quad \text{and } \theta_2 = \theta_1 + \theta_2 \text{ explicit!}
\[ e^{i\theta} = \cos \theta + i \sin \theta \]

a. \[ e^{-i\theta} = \cos(-\theta) + i \sin(-\theta) = \cos \theta - i \sin \theta \]

c. \[ \sin \theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) \]

d. \[ \cos \theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \]

**Note:**

Notice the formal similarity to the hyperbolic functions

\[ \cosh(x) = \frac{e^x + e^{-x}}{2} \]
\[ \sinh(x) = \frac{e^x - e^{-x}}{2} \]

Cosine and sine are \( \cosh \) and \( \sinh \) with imaginary arguments. Considered in terms of complex \( i^2 \), \( \cosh \) and \( \cos \) are just 2 parts of the same function...
N = 50 \text{ cm/s}; \quad T = 6 \text{ sec}; \quad t = 0, \theta = 30^\circ = \frac{\pi}{6} \text{ rad}

\omega = \frac{2\pi}{T} = \frac{2\pi}{6} = \frac{\pi}{3} \text{ rad/s}

w = \frac{v}{r} \Rightarrow r = \frac{v}{\omega} = \frac{150}{\pi} \text{ cm}

\quad a. \quad x = r \cos (\omega t + \phi) \quad \text{becomes} \quad (\text{I use } r = A)

\quad \quad \quad = \frac{150}{\pi} \cos \left( \frac{\pi}{3} t + \frac{\pi}{6} \right)

\quad b. \quad x = \frac{150}{\pi} \cos \left( \frac{\pi}{3} t + \frac{\pi}{6} \right) \text{ cm}

\quad \quad \quad \dot{x} = \frac{dx}{dt} = -50 \sin \left( \frac{\pi}{3} t + \frac{\pi}{6} \right) \text{ cm/s}

\quad \quad \quad \ddot{x} = \frac{d^2x}{dt^2} = -50 \frac{\pi}{3} \cos \left( \frac{\pi}{3} t + \frac{\pi}{6} \right) \text{ cm/s}^2

\quad \text{At } t = 2 \text{ sec, the arguments are } \frac{2\pi}{3} + \frac{\pi}{6} = \frac{5\pi}{6}

\quad \quad \quad \therefore \quad x = \frac{150}{\pi} \cos \left( \frac{5\pi}{6} \right) = \frac{150}{\pi} \left[ -\frac{\sqrt{3}}{2} \right] = -\frac{75\sqrt{3}}{\pi} \text{ cm}

\quad \quad \quad \dot{x} = -50 \sin \left( \frac{5\pi}{6} \right) = -50 \cdot \frac{1}{2} = -25 \text{ cm/s}

\quad \quad \quad \ddot{x} = -50 \frac{\pi}{3} \left[ -\frac{\sqrt{3}}{2} \right] = \frac{25\pi}{3} \text{ cm/s}^2
\(y_1 = A \cos(10\pi t) = A \cos(\omega_t)\)
\(y_2 = A \cos(12\pi t) = A \cos(2\omega_t)\)

Note we can describe the 2 frequencies here as

\[\begin{align*}
W_0 &= \omega_0 - \Delta \omega = 10\pi \\
W_1 &= \omega_0 + \Delta \omega = 12\pi
\end{align*}\]

for \(\omega_0 = 11\pi\) and \(\Delta \omega = \pi\)

\(\therefore \omega_0 = \frac{1}{2}(\omega_0 + \omega_2), \quad \Delta \omega = \frac{1}{2}(\omega_2 - \omega_1)\)

Thus

\[y_1 + y_2 = A\left[\cos(\omega_0 t) + \cos(\omega_1 t)\right]\]

\[= A\left\{\cos[(\omega_0 - \omega_1)t] + \cos[(\omega_0 + \omega_1)t]\right\}\]

After expanding \(\cos(\alpha \pm \beta)\) and simplifying:

\[y_{12} = y_1 + y_2 = 2A \cos(\Delta \omega t) \cos(\omega_0 t) = 2A \cos(\pi t) \cos(11\pi t)\]

to draw, note that \(\Delta \omega\) is low frequency compared to \(\omega_0\). So we may think of the function above as \(\cos(\omega_0 t)\) with a slowly varying amplitude given by \(2A \cos(\Delta \omega t)\), the "function" has \(\omega_0 = 11\pi\) \(\Rightarrow\) \(T = 2\frac{\pi}{\omega_0} = 2\frac{2\pi}{11}\) \(\Rightarrow\) \(f = \frac{11}{2}\) hertz. So, in 2 seconds, the amplitude goes through 11 cycles and the "oscillation function" goes through 11 cycles:

the best period is \(\frac{1}{2}\) the period if the amplitude = 1 sec.
6) \[ x(t) = A \cos(wt + \phi) = 10 \cos(wt) + 17 \sin(wt) \]

\[ = 10 \cos(wt) + 17 \cos(wt - \pi/2) \]

Using phasors, we see that the sum of these two cosines is the sum of the x-projections of 2 rotating vectors separated by a phase difference of \( \pi/2 \):

![Diagram of vectors](image)

To add x coordinates we could just as well add the vectors, they are at right angles, so its easy.

\[ A = \sqrt{10^2 + 17^2} \approx 19.7 \]

\[ \phi = \tan \left( \frac{17}{10} \right) \approx -59.5^\circ \]

\[ x(t) = 19.7 \cos(wt - 59.5^\circ) \]